Problem Set 4

Student handout

Orders, interpolation, and Fourier inversion

Optional enrichment | 6 problems | 80 points + 10 challenge points | Suggested time: 3–4 hours

Preparation. Course packet Sections 5.10–5.11 and polynomial interpolation from Chapter 4. Use $i^2 = -1$ for complex arithmetic; no calculus or infinite Fourier series is needed.

Name: _____ **Due:** _____

Instructions. Write on separate paper and use exact arithmetic. You may use poset Möbius inversion and polynomial CRT, except when a problem asks you to establish the relevant formula. Problems 1–3 concern order and counting; Problems 4–6 concern polynomial evaluation and its inverse.

Conventions. For a finite poset P, $\mu_P(x,x) = 1$ and $\mu_P(x,y) = -\sum_{x \le z < y} \mu_P(x,z)$ when $x < y$. For a primitive nth root ω, our Fourier convention is

$$y_j = \sum_{k=0}^{n-1} c_k \omega^{jk}, \qquad c_k = \frac{1}{n} \sum_{j=0}^{n-1} y_j \omega^{-jk}.$$

Over a finite field, $1/n$ means a multiplicative inverse in that field.

I. Recover exact counts from cumulative counts

1. Four incomparable middle elements [12 points]

Let $P = \{0, a, b, c, d, 1\}$, with $0 < x < 1$ for each $x \in \{a, b, c, d\}$ and no comparisons between distinct middle elements.

(a) Draw the Hasse diagram and list the intervals with exactly two elements. (3)

(b) Compute $\mu_P(0,x), \mu_P(x,1)$ for the middle elements and $\mu_P(0,1)$. (4)

(c) Suppose $G(y) = \sum_{x \le y} F(x)$ and

y	0	a	b	c	d	1
$G(y)$	3	5	8	10	4	25

Recover all six values of F. Check $F(1)$ both by subtraction and by Möbius inversion. (5)

2. Membership patterns, including none [14 points]

A universe U has 50 elements, and three subsets satisfy

$$|A| = 24, \quad |B| = 22, \quad |C| = 18, \qquad |A \cap B| = 10, \quad |A \cap C| = 8, \quad |B \cap C| = 6, \qquad |A \cap B \cap C| = 3.$$

(a) For $T \subseteq \{A, B, C\}$, let $F(T)$ count elements belonging to exactly the sets in T. Let $G(S)$ count elements in every set in S, with $G(\varnothing) = 50$. Explain why $G(S) = \sum_{T \supseteq S} F(T)$ is an *upper* sum on the subset poset. (4)

(b) Use upper-sum Möbius inversion to recover all eight exact membership counts, including $F(\varnothing)$. State the formula used. (8)

(c) Compute $|A \cup B \cup C|$. Check that your eight counts are nonnegative and sum to 50. (2)

3. Why a partition coefficient can be six [14 points]

Let Π_4 be the partitions of $\{1,2,3,4\}$, ordered by refinement. Write $\hat{0} = 1|2|3|4$ and $\hat{1} = 1234$. Partitions have block-size types $1+1+1+1$, $2+1+1$, $2+2$, $3+1$, or 4.

(a) Count the partitions of each type. (4)

(b) Compute $\mu_{\Pi_4}(\hat{0}, \pi)$ for each type, ending with $\mu_{\Pi_4}(\hat{0}, \hat{1})$ by the recurrence. *Hint:* for type $2+2$, the interval is a product of two two-element chains; for type $3+1$, it is Π_3. (6)

(c) There are a colors. Let $F(\rho)$ count colorings of four labeled positions whose equality partition is exactly ρ, and let $G(\pi)$ count colorings constant on every block of π. Explain why $G(\pi) = a^{|\pi|} = \sum_{\rho \geq \pi} F(\rho)$, where $|\pi|$ counts blocks. Invert to count colorings with all four colors distinct. Give the result for $a = 5$ and compare with direct counting. (4)

II. Recover coefficients from values

4. Lagrange selectors become Fourier inversion [12 points]

Let $n \geq 2$ and let $\omega \in \mathbb{C}$ have order exactly n.

(a) Use a finite geometric sum to show, for every integer m,

$$\sum_{j=0}^{n-1} \omega^{jm} = \begin{cases} n, & n \mid m, \\ 0, & n \nmid m. \end{cases}$$

Explain where primitivity of ω enters. (4)

(b) Define $\ell_j(X) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-jk} X^k$. Show that $\ell_j(\omega^r) = 1$ for $r = j$ and 0 otherwise, for $0 \leq r, j < n$. Explain why this identifies the Lagrange basis. (4)

(c) If $f(X) = \sum_{k=0}^{n-1} c_k X^k$ and $y_j = f(\omega^j)$, expand $f = \sum_j y_j \ell_j$ to derive the inverse Fourier formula. Explain the role of the degree restriction. (4)

5. The same four-point transform in two fields [16 points]

Take $f(X) = 2 - X + 3X^3$, with coefficient vector $(2, -1, 0, 3)$. For each field below, compute the four evaluations and then recover the coefficients using the inverse transform, showing each inverse sum.

(a) Over $\mathbb{C}$, use $\omega = i$ and list values in the order $1, i, -1, -i$. (8)

(b) Over $\mathbb{F}_{13}$, prove that $\omega = 5$ has order 4. List values in the order $1, 5, 12, 8$. Find both ω^{-1} and 4^{-1} before applying the inverse formula. Interpret the recovered coefficient of X. (8)

6. Why Fourier multiplication wraps around [12 points]

Work over $\mathbb{C}$ with four evaluation points $1, i, -1, -i$. Let $f(X) = 1 + 2X + X^3$ and $g(X) = 2 + X^2$.

(a) Multiply ordinarily, then reduce modulo $X^4 - 1$ to find the unique remainder h of degree less than 4. (4)

(b) Compute the two value vectors, multiply them entry by entry, and invert to recover the coefficient vector of h. (4)

(c) Write the cyclic convolution formula for the coefficient of X^r in h, using coefficient vectors $c = (1, 2, 0, 1)$ and $d = (2, 0, 1, 0)$. Explain why the X^5 term contributes to X, and how polynomial CRT justifies multiplication of values. (4)

Optional challenge: recover the ordinary product by adding zeros [10 extra points]

Use the same f and g as in Problem 6.

(a) What is the smallest transform length N for which the remainder of fg modulo $X^N - 1$ must equal the ordinary product? Explain why length 8 also works. (4)

(b) Work now over $\mathbb{F}_{17}$. Show that 2 has order 8 and find the inverse normalization factor 8^{-1}. (3)

(c) Append zeros to make both coefficient vectors length 8. Explain why evaluating at the powers of 2, multiplying values, and inverting recovers the ordinary product over $\mathbb{F}_{17}$. Give the resulting length-8 coefficient vector; you need not compute the eight-point transforms explicitly. (3)